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DESYNCHRONIZATION OF LINEAR SYSTEMS

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Introduction

The studies of dynamics inherent in control systems which incorporate sampled data elements such as extrapolators, keys, memory elements, etc. are usually reduced to the analysis of difference equations. However, this approach tends to disregard the inevitable small desynchronization of times when sampled elements are connected. In some systems this desynchronization does not influence the stability; otheres may be destabilized by any infinitesimal desynchronization; finally, some unstable synchronized systems can become asymptotically stable following the introduction of any infinitesimal desynchronization. These facts account for many phenomena observed in the engineering practice that looked enigmatic when the mathematical models ignored small desynchronization. In designing the controllers the stability can be achieved by introduction of lags into the system.

The article consists of three sections. Section 1 contains formulations of basic results in the stability theory of linear desynchronized systems. Section 2 is a discussion of some ways to extend the results. Section 3 provides examples.

All the assertions are given without proof. The proofs will be published in *Avtomatika i Telemekhanika* in 1983–1984.

1. Basic results

Systems W are studied whose dynamics in continuous time are described by the equations

$$x_i\left(T_i^{n+1}\right) = \sum_{j=1}^{N} a_{ij} x_j\left(T_i^n\right), \quad i = 1, 2, \ldots, N. \tag{1.1}$$

Every variable x_i is either a scalar or a vector of a dimension d_i. The function $x_i(t)$ over every interval $T_i^n < t \leqslant T_i^{n+1}$ of time variation between the instances T_i^n and T_i^{n+1} takes on a constant value. In the case of scalar variables the symbol a_{ij} denotes numbers and in the case of vector variables it denotes rectangular or square matrices of associated dimensions.

1.1. Synchronized systems

Below A denotes a matrix with elements a_{ij} (which can be matrix blocks), the order of the matrix A is $d_1 + d_2 + \cdots + d_N$. The vector-function $\{x_1(t), x_2(t), \ldots, x_N(t)\}$ is denoted as $x(t)$. The spectral radius of the square matrix B (i.e. the largest magnitude of its eigenvalues) is denoted by $r(B)$.

The system (1.1) is referred to as synchronized if $T_1^n = T_2^n = \cdots = T_N^n$ for every n. The dynamics of the synchronized system (1.1) is described by a vector-valued difference equation

$$x^{n+1} = Ax^n, \tag{1.2}$$

where $x^n = x(T^n)$ and $T^n = T_1^n (= T_2^n = \cdots = T_N^n)$. Consequently, by virtue of the standard theorems [1], a synchronized system is asymptotically stable iff $r(A) < 1$.

The system (1.1) is referred to as desynchronized if it is not synchronized. If the system (1.1) is synchronized then it is usually assumed that $T_i^n =$

nh $(i=1,\ldots,N)$ where $h>0$. If $T_i^n = nh + \tau_i$ where not all the numbers τ_i are equal, then the system (1.1) is referred to as phase-desynchronized and the numbers τ_i as phase shifts. If $T_i^n = n(h+\delta_i)$ where not all δ_i are equal, then the system (1.1) is referred to as frequency-desynchronized and the numbers δ_i as frequency shifts.

1.2. Systems insensitive to desynchronizations

The system (1.1) is referred to as regular if $T_i^n \to \infty$ for each fixed $i = 1, 2, \ldots, N$. The class of regular systems is very wide, it includes synchronized systems and phase- and frequency-desynchronized systems.

Theorem 1. *Let the system* (1.1) *be regular, the matrix A symmetrical, and* $r(A) < 1$. *Then the system* (1.1) *is asymptotically stable.*

Let us denote $|A|$ the matrix whose numerical elements are equal to the absolute value of the associated elements of the matrix A.

Theorem 2. *Let the system* (1.1) *be regular and* $r(|A|) < 1$. *Then the system* (1.1) *is asymptotically stable.*

In the conditions of Theorem 1 and 2 the system (1.1) is asymptotically stable no matter whether it is synchronized or desynchronized; the system (1.1) is asymptotically stable with large as well as small differences of switching times. Efficiency of Theorems 1 and 2 is made stronger by the possibility to apply various well-known methods to estimate the spectral radii [1,2] rather then to solve the associated characteristic equations. For example, if A is numerical matrix then the inequality $r(|A|) < 1$ is certainly true if one of the following inequalities holds:

$$\max_i \sum_j |a_{ij}| < 1, \quad \max_j \sum_i |a_{ij}| < 1, \quad \sum_{i,j} a_{ij}^2 < 1. \tag{1.3}$$

In one important case Theorem 2 can be inverted.

Theorem 3. *Let A be a numerical matrix and* $a_{ij} \geqslant 0$. *Let the regular system (1.1) be asymptotically stable. Then* $r(A) < 1$.

1.3. Phase desynchronization

Denote by $A_{i_1 i_2 \ldots i_k}$ where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$ a matrix which is obtained from an identity (numerical or block) matrix by replacing rows enumerated as $i_1, i_2, \ldots, i_k$ by the associated rows of the matrix A. For instance,

$$A_i = \begin{pmatrix} I & 0 & \ldots & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ldots & \\ a_{i1} & a_{i2} & \ldots & a_{ii} & \ldots & a_{iN} \\ \ldots & \ldots & \ldots & & \ldots & \\ 0 & 0 & \ldots & 0 & \ldots & I \end{pmatrix}$$

Let us assume at this point that the system (1.1) is phase-desynchronized. Let $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_N$; this positioning of the numbers τ_i can always be achieved by changing the enumeration of the components in (1.1).

Theorem 4. *Let the system* (1.1) *be phase-desynchronized and* $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_N < h$. *Then the system* (1.1) *is asymptotically stable iff* $r(C) < 1$ *where*

$$C = A_N A_{N-1} \cdots A_1. \tag{1.4}$$

If $r(C) = 1$ *and the order and multiplicity of every eigenvalue of the matrix* (1.4) *on the circumference* $|\lambda| = 1$ *coincide, then the system* (1.1) *is neutrally stable. In the remaining cases the system* (1.1) *is unstable.*

If some of the phase shifts τ_i coincide, then in constructing the matrix (1.4) every product $A_{m+k} \cdot \ldots \cdot A_m$ that is associated with a group of identical shifts $(\tau_{m-1} < \tau_m = \tau_{m+1} = \cdots = \tau_{m+k} < \tau_{m+k+1})$ must be replaced by one matrix $A_{m\ldots(m+k)}$. Following this all assertions of Theorem 4 remain valid.

In Theorem 4 the values of phase shifts are unimportant, only their sequence being essential.

In a general case the matrix C is different from the matrix A. What is important is that either can be stable or unstable independently of the other (see Example 3.3). From this remark and Theorem 4 follows the desirability of artificially introduced phase shifts in some cases.

The spectrum of the matrix (1.4) can be investigated without constructing the matrix.

Theorem 5. *Let a_{ij} be numbers. Let $0 \leqslant \tau_1 < \tau_2 < \ldots < \tau_N < h$. Then the eigenvalues of the matrix* (1.4) *coincide with the roots of the algebraical equation*

$$\det \begin{pmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1N} \\ \lambda a_{21} & a_{22}-\lambda & \ldots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{N1} & \lambda a_{N2} & \ldots & a_{NN}-\lambda \end{pmatrix} = 0. \quad (1.5)$$

If some of the shifts τ_i coincide (this includes situations where the elements a_{ij} are matrix blocks), then in (1.5) only those subdiagonal elements a_{ij} are multiplied by λ whose subscripts i and j satisfy the condition $\tau_i \neq \tau_j$.

1.4. Frequency desynchronization

Let us assume that the system (1.1) is frequency-desynchronized. We are interested only in those cases where Theorems 1 or 2 cannot be applied. Let

$$h_1 = h + \delta_1, h_2 = h + \delta_2, \ldots, h_N = h + \delta_N. \quad (1.6)$$

If the periods (1.6) are commensurable, then a minimal $h^* > 0$ can be found which is multiple of all h_i; and then a matrix C^* of transition from the initial states $x(0)$ to the states $x(h^*) = C^* x(0)$ can be written. In terms of the properties of the matrix C^* assertions on stability of the system (1.1) analogous with Theorem 4 are easily formulated. We are unaware of the existence of any simple (akin with Theorem 5) rules for computation of the eigenvalues of the matrix C^* directly from the matrix A, actual construction of the matrix C^* would be unwieldy, as a rule. Furthermore, the above reasoning is inapplicable if some of the periods (1.6) are not commensurable.

Let us choose the number $H > 0$ and denote by $D_n(H)$ a matrix of transition from the states $x(nH)$ to the states $x(nH + H) = D_n(H)x(nH)$. In the matrix sequence $D_n(H)$ there are only a finite number $F_1, F_2, \ldots, F_m$ of different matrices. Let $\mu(j, k)$ be the number of matrices in the totality $D_1(H), \ldots, D_k(H)$ which coincide with F_j. There exist the limits

$$p_j = \lim_{k \to \infty} k^{-1} \mu(j, k), \quad j = 1, \ldots, m, \quad (1.7)$$

moreover, $p_1 + p_2 + \cdots + p_m = 1$. For every $j = 1, 2, \ldots, m$ the number p_j can be treated as the mean frequency of the matrix F_j occurring in the matrix sequence $D_k(H)$. With small N the numbers (1.7) can be easily computed or estimated.

Introduce the notation

$$q(H) = \|F_1\|^{p_1} \|F_2\|^{p_2} \cdots \|F_m\|^{p_m}, \quad (1.8)$$

$$d(H) = |\det F_1|^{p_1} |\det F_2|^{p_2} \cdots |\det F_m|^{p_m}. \quad (1.9)$$

Formula (1.9) determines the value of $d(H)$ in the case when all the numbers $\det F_j$ are nonzero. If at least one of them vanishes, then let us make $d(H) = 0$.

Theorem 6. *If $q(H) < 1$, then the system* (1.1) *is asymptotically stable; if $d(H) > 1$, then it is unstable.*

The value of $q(H)$ does not increase with multiplication of H: $q(nH) \leqslant q(H)$. Therefore, if in verifying the condition of Theorem 6 the value of $q(H)$ with some H is found to exceed 1, then H can be multiplied by an integer and then the condition of the theorem can be verified with this new value of H. It should be borne in mind, however, that with H or N increasing, the number of matrices F_i rapidly grows. Therefore, Theorem 6 is sufficiently active only with small H and N. An example of using Theorem 6 is given in Example 3.4.

The value of $d(H)$ is in fact independent of H. Therefore the verification of the condition $d(H) > 1$ for a single value of H would be sufficient.

Let $G_1, \ldots, G_{N!}$ denote all the products $A_{i_1} A_{i_2} \cdots A_{i_N}$ where $(i_1, i_2, \ldots, i_N)$ are permutation of the numbers $(1, 2, \ldots, N)$.

Theorem 7. *If $\|G_1\| \cdot \ldots \cdot \|G_{N!}\| < 1$, then for all sufficiently small δ_i and incommensurable in totality periods* (1.6) *the system* (1.1) *is asymptotically stable. If* $\det A \neq 0$ *and* $|\det a_{11}| \cdot |\det a_{22}| \cdot \cdots \cdot$

$|\det a_{NN}| > 1$, *then for all sufficiently small* δ_i *and incommensurable in totality periods* (1.6) *the system* (1.1) *is unstable.*

1.5. Two-component system

A system

$$x_1(nh+h) = a_{11}x_1(nh) + a_{12}x_2(nh),$$
$$x_2[(n+1)(h+\delta)] = a_{21}x_1(nh+n\delta) + a_{22}x(nh+n\delta) \qquad (1.10)$$

will be referred to as two-component. The dimensions of the vectors x_1 and x_2 in the system (1.10) are equal to d_1 and d_2, respectively.

To study the system (1.10), let us construct a sequence of five-tuples

$$\{\lambda_i, S_i, T_i, \hat{S}_i, \hat{T}_i\}, \quad i=0,1,\dots \qquad (1.11)$$

where λ_i are numbers in the interval [0,1] and S_i, T_i, $\hat{S}_i$, $\hat{T}_i$ are square matrices of the order d_1+d_2. Let

$$\lambda_0 = \frac{h}{2h+\delta}, \quad S_0 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & I \end{pmatrix},$$
$$T_0 = \begin{pmatrix} I & 0 \\ a_{21} & a_{22} \end{pmatrix}, \quad \hat{S}_0 = A, \quad \hat{T}_0 = I.$$

If the five-tuple $\{\lambda_j, S_j, T_j, \hat{S}_j, \hat{T}_j\}$ has been obtained, then for $0 < \lambda_j \leqslant \frac{1}{2}$ let us set

$$\lambda_{j+1} = 2 + n_j - \lambda_j^{-1}, \quad S_{j+1} = T_j S_j^{n_j+1},$$
$$T_{j+1} = T_j S_j^{n_j}, \quad \hat{S}_{j+1} = \hat{T}_j \hat{S}_j S_j^{n_j},$$
$$\hat{T}_{j+1} = T_j S_j^{n_j} \qquad (1.12)$$

where $n_j = -[1-\lambda_j^{-1}] - 1$ (here $[\nu]$ is the greatest integer not exceeding ν). For $\frac{1}{2} < \lambda_j < 1$ let

$$\lambda_{j+1} = (1-\lambda_j)^{-1} - n_j - 1, \quad S_{j+1} = T_j^{n_j} S_j,$$
$$T_{j+1} = T_j^{n_j+1} S_j, \quad \hat{S}_{j+1} = T_j^{n_j} S_j,$$
$$\hat{T}_{j+1} = T_j^{n_j} \hat{T}_j \hat{S}_j \qquad (1.13)$$

where $n_j = -[1-(1-\lambda_j)^{-1}] - 1$.

Associate every five-tuple (1.11) with the number

$$R_i = \|S_i\|^{1-\lambda_i} \|T_i\|^{\lambda_i}. \qquad (1.14)$$

The construction of the sequence (1.11) is terminated either if for some i one of the equalities $\lambda_i = 0$ or $\lambda_i = 1$ holds or if $0 < \lambda_i < 1$ and $R_i < 1$.

Theorem 8. *Let* $\det A$, $\det a_{11}$, $\det a_{22} \neq 0$. *The system* (1.10) *is asymptotically stable iff for some* i *the five-tuple* (1.11) *has one of the following three properties: either* $\lambda_i = 0$ *and* $r(\tilde{S}_i) < 1$, *or* $\lambda_i = 1$ *and* $r(\hat{T}_i) < 1$, *or* $0 < \lambda_i < 1$ *and* $R_i < 1$.

In the cases where $\lambda_i = 0$ *or* $\lambda_i = 1$, *the periods* h *and* $h+\delta$ *are commensurable. If* $\lambda_i = 0$ *and* $r(\hat{S}_i) \geqslant 1$ *or* $\lambda_i = 1$ *and* $r(\hat{T}_i) \geqslant 1$, *then the system* (1.10) *is not asymptotically stable.*

What is important is that the algorithm of Theorem 8 rapidly determines, as a rule, whether the system (1.1) is stable. On the other hand, in computating formulae (1.12) and (1.13) the computation error dramatically increases (see Example 3.5). Thus, for the absolute error Δ_i of measuring the quantity λ_i the estimates are valid

$$(2+n_0)^2 \cdots (2+n_{i-1})^2 \Delta_0 \geqslant \Delta_i$$
$$\geqslant (1+n_0)^2 \cdots (1+n_{i-1})^2 \Delta_0. \qquad (1.15)$$

For the absolute error Δ_i^* of measuring of the elements of the matrices S_i, T_i, $\hat{S}_i$, $\hat{T}_i$ the estimate is true

$$\Delta_i^* \leqslant (2D)^i [M_0(2+n_0)] \dots [M_{i-1}(2+n_{i-1})] \Delta_0^*, \qquad (1.16)$$

where M_j is the greatest absolute value of the elements of the matrices S_j, T_j, $\hat{S}_j$, $\hat{T}_j$ and $D = d_1 + d_2$ is the dimension of the system (1.10). Consequently, with $\Delta_0 = 10^{-6}$ by virtue of (1.15) $\Delta_i \geqslant 2^{2i} \cdot 10^{-6}$ and the computation (1.12)–(1.13) makes no sense even with $i = 10$.

An example of using Theorem 8 is given in Example 3.6.

In the case where in (1.10) a_{ij} are numbers and the frequency shift δ is small, fairly general conditions for stability or unstability of the system (1.10) can be obtained directly in terms of elements of the matrix A.

Theorem 9. *If* $r(A_2A_1) < 1$, *then the system* (1.10) *is asymptotically stable for every, sufficiently small,*

$\delta \neq 0$. If $r(A_2A_1) > 1$ and either

$$|a_{11}a_{22}| > 1, \quad a_{11}a_{22} \neq a_{12}a_{21}, \tag{1.11}$$

or

$$\begin{aligned} &|a_{11}a_{22}| < 1, \\ &(a_{11} - a_{22})^2 + a_{12}a_{21}(1 + a_{11})(1 + a_{22}) \neq 0, \\ &(2a_{11}a_{22} - a_{12}a_{21})^2 - (a_{11} + a_{22}) \\ &\quad \times (a_{11} + a_{22} + a_{12}a_{21}) \\ &\quad \times (2a_{11}a_{22} - a_{12}a_{21}) + a_{11}a_{22}(a_{11} + a_{22})^2 \neq 0, \end{aligned} \tag{1.12}$$

holds, then the system (1.10) *is unstable for every, sufficiently small, $\delta \neq 0$.*

It follows from Theorem 9 (see Example 3.3) that the asymptotically stable synchronized system by introducing the frequency shifts as small as desired can become unstable and an unstable system can become asymptotically stable. As a result, introduction of artificial frequency shifts can serve a useful purpose in some cases.

2. Remarks

Remark 2.1. The dynamics of the system W described by the equations

$$x_i(T_i^{n+1}) = \sum_{j=1}^{N} a_{ij}x_j(T_i^n), \tag{2.1}$$

where

$$x_i(t) = \text{const} \quad \text{for } T_i^n < t \leqslant T_i^{n+1}, \tag{2.2}$$

can be described in equivalent terms

$$x_i(T_i^n + 0) = \sum_{j=1}^{N} a_{ij}x_j(T_i^n - 0), \tag{2.3}$$

where

$$x_i(t) = \text{const} \quad \text{for } T_i^n < t < T_i^{n+1}. \tag{2.4}$$

Here $x_i(T_i^n + 0)$ and $x_j(T_i^n - 0)$ denote right-hand and left-hand limits, respectively, at the point T_i^n. In (2.3) the value of the function $x_i(t)$ at switching times is unimportant.

A situation is described in Example 3.1 which leads to equations (2.1), (2.2). The dynamics of the system in Example 3.2 is described by the equations (2.1) where

$$x_i(t) = \text{const} \quad \text{for } T_i^n \leqslant t < T_i^{n+1}. \tag{2.5}$$

Equations (2.1), (2.5) are not generally equivalent with (2.1), (2.2). Consequently, for (2.1), (2.4) the above assertions need a modification to remain valid.

We do not know whether the studies of the system (2.1), (2.5) can be reduced to those of a system of the form (2.1), (2.2). One particular case where this reduction is possible is described below.

The system (2.1), (2.5) will be said to be completely desynchronized if no two of its components switch simultaneously.

Let the system (2.1), (2.5) be completely desynchronized. Assume that for every $i = 1, 2, \ldots, N$ and every n

$$y_i(t) = x_i(T_i^n) \quad \text{for } T_i^{n-1} < t \leqslant T_i^n, \tag{2.6}$$

$$z_i(t) = x_i(T_i^n) \quad \text{for } T_i^n < t \leqslant T_i^{n+1}. \tag{2.7}$$

Then

$$y_i(T_i^{n+1}) = a_{ii}y_i(T_i^n) + \sum_{j \neq i} a_{ij}z_j(T_i^n), \tag{2.8}$$

$$z_i(T_i^{n+1}) = y_i(T_i^n). \tag{2.9}$$

Consequently, the study of a completely desynchronized system (2.1), (2.5) is reducible to that of a system (2.6)–(2.9) which is a system of the form (2.1), (2.2).

Remark 2.2. Consider a linear system W whose dynamics in continuous time are described by equations, more general than (1.1),

$$x_i(T_i^{n+1}) = \sum_{j=1}^{N} a_{ij}x_j(T_i^n) + u_i(T_i^n), \tag{2.10}$$

which are different from (1.1) in that exogenous signals $u_i(t)$ are present.

Assume that for constant exogenous signals $u_i(t) \equiv u_i^*$ the system (2.10) has an equilibrium state $x_i(t) \equiv x_i^*$. This state is asymptotically stable (neutrally stable, unstable) iff so is the associated uniform system (1.1).

Remark 2.3. Let us consider a nonlinear system W whose dynamics are described in continuous time by the equations

$$x_i(T_i^{n+1}) = f_i[x_1(T_i^n), \ldots, x_N(T_i^n)], \tag{2.11}$$

where x_i are scalar variables. Let $x(t) \equiv x^*$ be the equilibrium state of the system (2.11), i.e. $f_i(x_1^*, \ldots, x_N^*) = x_i^*$. To study the stability of this state let us build a first approximation system:

$$z_i(T_i^{n+1}) = \sum_{j=1}^{N} f_{ij} z_j(T_i^n),$$

where $f_{ij} = (\partial f_i / \partial x_j)(x_1^*, \ldots, x_N^*)$.

The conventional theorems on stability in the first approximation are valid.

3. Examples

Example 3.1. Let us consider a multi-processor system W which consists of N processors $P_1, P_2, \ldots, P_N$ which exchange data through the buffer B (see Fig. 1). Assume that the working cycle of every processor P_i starts at specified times $T_i^1, T_i^2, \ldots, T_i^n, \ldots$ and consists of three phases.

Phase 1. Reception of the contents of the buffer B, the vector $x = \{x_1, x_2, \ldots, x_N\}$.

Phase 2. Computation of the new value of the ith component of the vector x according to the rule

$$x_{i\,\text{new}} = a_{i1}x_1 + \cdots + a_{iN}x_N + u_i(T_i^n). \tag{3.1}$$

Phase 3. Recording the resultant new value $x_{i\,\text{new}}$ in the appropriate location in the buffer B.

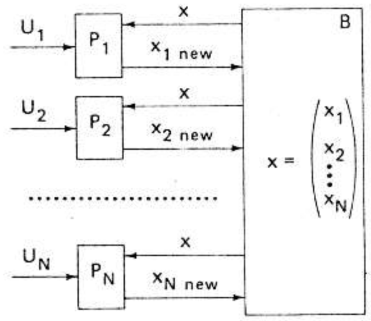

Fig. 1.

Let the duration of all the three phases in each processor be small in comparison with the time $T_i^{n+1} - T_i^n$ between two subsequent start ups of the processors, Fig. 2(a). Then the dynamics of the multi-processor system W are described by equations (2.1), (2.2) (and with $u_i(t) \equiv 0$, by equations (1.1)).

Example 3.2. Let in the system W of Example 3.1 the duration of the first and second phases in each processor be small in comparison with that of the interval between two subsequent start ups of the processor.

Let the computation of the new value $x_{i\,\text{new}}$ by formula (3.1) be slow in that the second phase takes up almost the entire interval between two subsequent start ups of the processor, Fig. 2(b). Then the dynamics of the system W are described by equations (2.1), (2.5).

Example 3.3. Table 1 represents the value of the spectral radii $r(A)$ and $r(A_2A_1)$ of four matrices A and associated matrices A_2A_1 and the results of checking the conditions (1.18) of Theorem 9.

The data represented in Table 1 shows that by introducing phase shifts or small frequency shifts

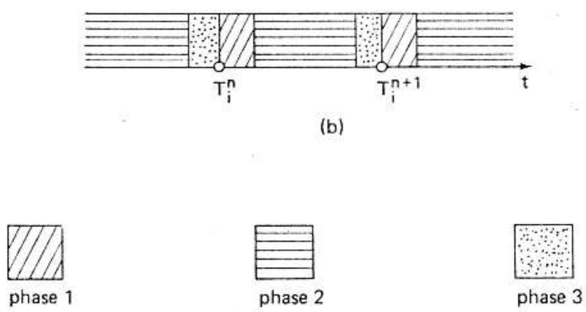

Fig. 2.

Table 1

A	−0.5	1.0	−0.5	−0.6	−0.5	−0.0	−0.5	2.0
	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
A_2A_1	−0.5	1.0	−0.5	−0.6	−0.5	0.0	−0.5	2.0
	0.25	−1.0	0.25	−0.2	0.25	−0.5	0.25	−1.5
$r(A)$	0.87		1.05			0.5		1.12
$r(A_2A_1)$	1.31		0.5			0.5		1.87
Condition (1.18) true?	yes		–			–		yes
Synchr. system	stable		unstable			stable		unstable
Desynchr. system	unstable		stable			stable		unstable

an asymptotically stable synchronized system can become unstable and an unstable synchronized system can become asymptotically stable.

Example 3.4. Consider a three-dimensional system (1.1) with the matrix

$$A = \begin{pmatrix} 0.6 & 0.4 & -0.4 \\ -0.4 & 0.6 & 0.4 \\ 0.4 & -0.4 & -0.1 \end{pmatrix}. \tag{3.2}$$

Let the system (1.1) with the matrix (3.2) be frequency-desynchronized, i.e. $T_1^n = nh_1$, $T_2^n = nh_2$ and $T_3^n = nh_3$ where

$$h_1 = 1, \quad h_2 = \sqrt{2}, \quad h_3 = \sqrt{3}. \tag{3.3}$$

Let us study the system (1.1), (3.2) with the aid of Theorem 6. Let us choose $H = h_3 = \sqrt{3}$. Then there are 10 matrices F_i:

$$F_1 = A_2A_1A = \begin{pmatrix} 0.040 & 0.640 & -0.040 \\ -0.096 & -0.056 & 0.216 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_1\| = 0.900;$

$$F_2 = A_2A_1A_3 = \begin{pmatrix} 0.440 & 0.560 & 0.040 \\ -0.016 & 0.216 & -0.056 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_2\| = 1.040;$

$$F_3 = A_1A_2A_3 = \begin{pmatrix} 0.344 & 0.336 & 0.024 \\ -0.240 & 0.440 & -0.040 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_3\| = 0.900;$

$$F_4 = A_2A_1A_1A_3 = \begin{pmatrix} 0.104 & 0.896 & 0.064 \\ 0.118 & 0.082 & -0.066 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_4\| = 1.064;$

$$F_5 = A_1A_1A_2A_3 = \begin{pmatrix} -0.050 & 0.538 & 0.038 \\ -0.240 & 0.440 & -0.040 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_5\| = 0.900;$

$$F_6 = A_2A_1A_2A_3 = \begin{pmatrix} 0.344 & 0.336 & 0.024 \\ -0.122 & -0.030 & -0.074 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_6\| = 0.900;$

$$F_7 = A_1A_2A_1A_3 = \begin{pmatrix} 0.098 & 0.582 & 0.042 \\ -0.016 & 0.216 & -0.056 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_7\| = 0.900;$

$$F_8 = A_2A_1A_2A_1A_3 = \begin{pmatrix} 0.098 & 0.582 & 0.042 \\ 0.111 & -0.263 & -0.090 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_8\| = 0.900;$

$$F_9 = A_2A_1A_1A_2A_3 = \begin{pmatrix} -0.050 & 0.538 & 0.038 \\ 0.036 & -0.111 & -0.079 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_9\| = 0.900;$

$$F_{10} = A_1A_2A_1A_2A_3 = \begin{pmatrix} -0.002 & 0.349 & 0.025 \\ -0.122 & -0.030 & -0.074 \\ 0.400 & -0.400 & -0.100 \end{pmatrix},$$

$\|F_{10}\| = 0.900.$

$$F_0 = A_2 A_1 A_1 A_2 A_3$$

$$\sqrt{3} > x_2 + \sqrt{2} > x_1 + 1 > x_1 > x_2 > 0$$

$$x_1 + 2 > \sqrt{3},\ x_2 + 2\sqrt{2} > \sqrt{3}$$

Fig. 3.

Here the norm of the matrix F is generated by the norm $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ in the space $\mathbb{R}^3$ and, consequently, is dictated by the formula $\|F\| = \max_i \Sigma_j |f_{ij}|$.

What remains to do is to find the mean frequencies p_i of finding matrices F_i. This can be done directly by using formulae (1.7). There is, however, an alternative technique.

Every matrix F_i is uniquely represented in the form $F_i = Q_1 Q_2 \cdots Q_k$ where each Q_j is one of the matrices A_1, A_2, A_3, A. Associate matrix F_i that does not have a matrix A as a comultiplier with inequality system. The first on the right matrix of the form A_1 is associated with the symbol x_1, the second on the right matrix of the form A_1 is associated with the symbol $x_1 + h_1$, Fig. 3, etc. The first on the right matrix of the form A_2 is associated with the symbol x_2, the second on the right matrix of the form A_2 is associated with the symbol $x_2 + h_2$, etc. The resultant sequence of symbols will be arranged as the associated matrices are arranged in the expression for F_i. Insert between two neighbouring symbols $x_1 + kh_1$ or $x_2 + lh_2$ the inequality sign '$>$'. Add on the right of the resultant chain of inequalities the inequality '> 0' and on the left, the inequality '$H >$', Fig. 3.

Add two more inequalities: $x_1 + m_1 h_1 > H$ and $x_2 + m_2 h_2 > H$ where m_j ($j = 1, 2$) is the number of matrices of the form A_j in the expression for F_i.

The resultant inequality set will be referred to as the determining set of relations for the matrix F_i.

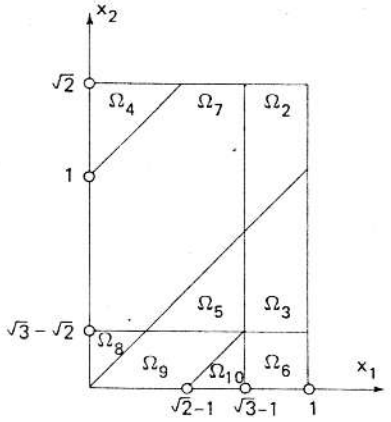

Fig. 4.

The determining set of relations for the matrix identifies in the rectangle $0 < x_1 < h_1$, $0 < x_2 < h_2$ a certain region Ω_i, Fig. 4.

Lemma 3.1. *If the numbers h_1, h_2 and $H = h_3$ ($h_3 > h_1, h_2$) are incommensurable in totality, then $p_i = h_1^{-1} h_2^{-1}$ mes Ω_i for matrices F_i which have a determining set of relations and $p_i = 0$ for matrices F_i which have not a determining set of relations.*

In a similar way a determining set of relations is obtained when $H \neq h_3$ but $H > h_1, h_2, h_3$. Here every matrix A_3 must also be associated with a symbol $x_3 + kh_3$. In this case the regions Ω_i are considered in a parallelepiped $0 < x_1 < h_1$, $0 < x_2 < h_2$, $0 < x_3 < h_3$ and the numbers p_i are determined by the equalities $p_i = (h_1 h_2 h_3)^{-1}$ mes Ω_i.

The described way of determining p_i is also applicable to systems whose dimension is higher than three. It is needed only that the periods $h_1, h_2, \ldots, h_N$ are incommensurable in totality.

Using Lemma 3.1. and Fig. 4. we have $p_1 = 0$, $p_2 = p_3 = 0.104$, $p_4 = p_5 = 0.061$, $p_6 = 0.060$, $p_7 =$

Table 2

i	0	1	2	3	4
λ_i^*	0.414214	0.585786	0.414214	0.585786	0.414214
λ_i	0.414214	0.585787	0.414216	0.585798	0.414282
i	5	6	7	8	9
λ_i^*	0.585786	0.414214	0.585786	0.414214	0.585786
λ_i	0.586187	0.416553	0.599342	0.495896	0.983449

Table 3

i	λ_i	n_i	S_i		T_i		$\|S_i\|$	$\|T_i\|$	R_i
0	0.41	1	−0.50	1.00	1.00	0.00	1.50	1.00	1.27
			0.00	1.00	−0.50	−0.50			
1	0.59	1	0.25	0.50	−0.50	1.00	0.88	1.50	1.20
			−0.13	−0.75	0.25	−1.00			
2	0.41	1	−0.25	−1.00	0.31	1.37	1.25	1.69	1.42
			0.19	0.88	−0.25	−1.12			
3	0.59	1	0.12	0.60	0.18	0.89	0.72	1.07	0.91
			−0.10	−0.49	−0.15	−0.73			

0.416, $p_8 = p_{10} = 0.036$, $\bar{p}_9 = 0.093$. Hence $q(H) = 0.923$. Consequently, by virtue of Theorem 6 the system (1.1), (3.2), (3.3) is asymptotically stable.

Example 3.5. Let $\lambda_0 = \sqrt{2} - 1$. In the second row of Table 2 the theoretical values of the numbers λ_i obtained by formulae (1.12), (1.13) are given with an accuracy of six significant digits. These theoretical values λ_i^* of λ_i have the form $\lambda_{2k}^* = \sqrt{2} - 1$ and $\lambda_{2k+1}^* = 2 + \sqrt{2}$. The third row shows the values of λ_i obtained in computation with the single precision by HP-2100A computer.

Example 3.6. Let us take up a frequency-desynchronized two-dimensional system (1.10) with the matrix

$$A = \begin{pmatrix} -0.5 & 1.0 \\ -0.5 & -0.5 \end{pmatrix} \tag{3.4}$$

and switching periods

$$h = 1, \quad h + \delta = \sqrt{2}. \tag{3.5}$$

The results of the computation by formulae (1.12) and (1.13) are summarized in the Table 3.

By virtue of Theorem 8 the system (1.10), (3.4), (3.5) is asymptotically stable. It is interesting to note that the synchronized system (1.10) with the matrix (3.4) is asymptotically stable. With small frequency shifts it becomes unstable (see Example 3.3), and with a large frequency shift ($h = 1$, $h + \delta = \sqrt{2}$) it again becomes asymptotically stable.

References

[1] F.R. Gantmakher, *The Theory of Matrices* (Nauka, Moscow, 1967) (in Russian).

[2] M.A. Krasnosel'skii, G.M. Vainicko et al., *Approximate Solution of Operator Equations* (Nauka, Moscow, 1969) (in Russian).